6.1: The Set of Rational Numbers

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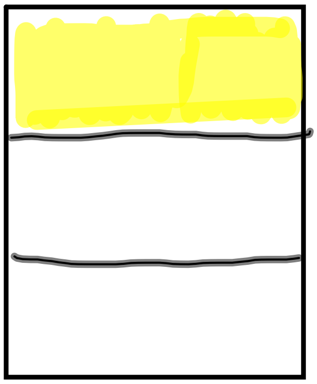
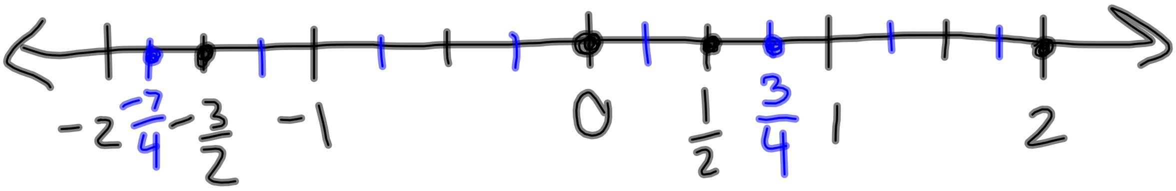
Definition: The rational numbers are all numbers of the form $\frac{a}{b}$, where a and b are integers with $b \neq 0$. We call a the numerator and b the denominator. We usually refer to these numbers in elementary school as fractions.

Example: Draw a figure to represent $\frac{1}{2}$ and $\frac{1}{3}$.



Example: Draw the points $\cancel{\frac{-3}{2}}$, $\cancel{0}$, $\cancel{\frac{3}{4}}$, $\cancel{2}$, $\cancel{\frac{-7}{4}}$, and $\cancel{\frac{1}{2}}$ on a number line.



6.1 Completed Notes

Definition: In the fraction $\frac{a}{b}$, if $|a| < |b|$, we call it a proper fraction. If $|a| \geq |b|$, we call it an improper fraction.

Example: List some proper and improper fractions.

Proper:

$\frac{1}{3}, \frac{-1}{2}, \frac{-1}{4}, \frac{3}{4}$

Improper:

$\frac{3}{3}, \frac{5}{3}, \frac{-7}{2}, \frac{-4}{4}$

Question: Is every integer a rational number? $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

Yes, $3 = \frac{3}{1}, -4 = \frac{-4}{1}, \ldots$

Definition: Two fractions that represent the same rational number are known as equivalent fractions.

Example: Find fractions that are equivalent to $\frac{1}{2}$ by folding paper.

Found that $\frac{1}{2} = \frac{2}{4} = \frac{4}{8} = \frac{3}{6}$

6.1 Completed Notes

Fundamental Law of Fractions: If $\frac{a}{b}$ is any fraction and n is a nonzero integer, then $\frac{a}{b} = \frac{an}{bn}$.

Example: Show that $\frac{^-7}{2} = \frac{7}{^-2}$.

$$\frac{^-7}{2} = \frac{^-7 \cdot {^-1}}{2 \cdot {^-1}} = \frac{7}{^-2}$$

Example: Find a value for x such that $\frac{3}{12} = \frac{x}{72}$.

$$\frac{3}{12} = \frac{3 \cdot 6}{12 \cdot 6} = \frac{18}{72} \quad \boxed{x = 18}$$

Definition: A rational number $\frac{a}{b}$ is said to be in simplest form if $b > 0$ and $\gcd(a,b) = 1$.

Example: Simplify the fraction $\frac{45}{300}$ by using the GCD.

$45 = 3^2 \cdot 5^1$

$300 = 2^2 \cdot 3^1 \cdot 5^2$

$GCD(45, 300) = 3 \cdot 5 = 15$

$$\frac{45}{300} = \frac{3 \cdot 15}{20 \cdot 15} = \boxed{\frac{3}{20}}$$

6.1 Completed Notes

Equality of Fractions: Show that $\frac{10}{16} = \frac{15}{24}$.

① $\frac{10}{16} = \frac{5\cdot 2}{8\cdot 2} = \boxed{\frac{5}{8}}$ $\quad \frac{15}{24} = \frac{5\cdot 3}{8\cdot 3} = \boxed{\frac{15}{8}}$ (reduce)

② $\frac{10}{16} = \frac{10\cdot 3}{16\cdot 3} = \boxed{\frac{30}{48}}$ $\quad \frac{15}{24} = \frac{15\cdot 2}{24\cdot 2} = \boxed{\frac{30}{48}}$ (LCM)

③ $\frac{10}{16} = \frac{10\cdot 24}{16\cdot 24} = \boxed{\frac{240}{384}}$ $\quad \frac{15}{24} = \frac{15\cdot 16}{24\cdot 16} = \boxed{\frac{240}{384}}$ (product)

④ 240 $\quad$ 240

$\frac{10}{16} = \frac{15}{24}$ (cross-multiply)

Theorem: Two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent if and only if $ad = bc$.
That is, we can cross multiply to check these.

Proof:

$$\frac{a}{b} = \frac{a\cdot d}{b\cdot d} = \frac{ad}{bd}$$

$$\frac{c}{d} = \frac{c\cdot b}{d\cdot b} = \frac{bc}{bd}$$

$$\frac{a}{b} = \frac{c}{d}$$

$$\Updownarrow$$

$$\frac{ad}{bd} = \frac{bc}{bd}$$

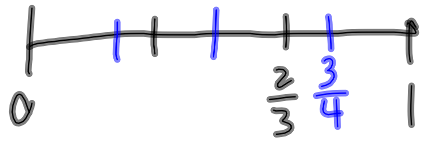
$$\Updownarrow$$

$$ad = bc$$

6.1 Completed Notes

Theorem: If a, b, and c are integers with $b > 0$, then $\frac{a}{b} > \frac{c}{b}$ if and only if $a > c$.

Example: Show that $\frac{9}{12} > \frac{6}{9}$.



(1) $\frac{9}{12} = \frac{3 \cdot 3}{4 \cdot 3} = \frac{3}{4}$ $\qquad$ $\frac{6}{9} = \frac{2 \cdot 3}{3 \cdot 3} = \frac{2}{3}$ (reduce)

(2) $\frac{9}{12} = \frac{9 \cdot 3}{12 \cdot 3} = \frac{27}{36}$ $\qquad$ $\frac{6}{9} = \frac{6 \cdot 4}{9 \cdot 4} = \frac{24}{36}$

(3) $\frac{9}{12} = \frac{9 \cdot 9}{12 \cdot 9} = \frac{81}{108}$ $\qquad$ $\frac{6}{9} = \frac{6 \cdot 12}{9 \cdot 12} = \frac{72}{108}$

(4) $9 \cdot 9 > 6 \cdot 12$

$81 > 72$

$\overset{81}{\frac{9}{12}} \bigcirc \overset{72}{\frac{6}{9}}$

Theorem: If a, b, c and d are integers with $b, d > 0$, then $\frac{a}{b} > \frac{c}{d}$ (<) if and only if $ad > bc$ (<).

Proof:

$$\frac{a}{b} = \frac{a \cdot d}{b \cdot d} = \frac{ad}{bd}$$

$$\frac{c}{d} = \frac{c \cdot b}{d \cdot b} = \frac{bc}{bd}$$

$$\frac{a}{b} \overset{<}{>} \frac{c}{d}$$

$$\Updownarrow$$

$$\frac{ad}{bd} \overset{<}{>} \frac{bc}{bd}$$

$$\Updownarrow$$

$$ad \overset{<}{>} bc$$